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Quasi-periodic oscillations on a nonlinear transient

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Abstract

The time-changing spectrum of a nonlinear transient is analyzed through the Gabor transform technique. Clusters of evenly spaced lines appear on spectrograms of the numerically computed oscillations. They can be explained by means of a simple model describing the dynamics of the energy exchanges between the external oscillating force and the nonlinear system. The resulting amplitude and frequency modulations are shown to produce the spectral line structures. Frequencies incommensurable with other present oscillations can be generated by the nonlinear system.

1. Introduction

Most literature on nonlinear oscillations [1–5] describes the steady state motion in detail, while the transients receive a qualitative description, typically an outline of the trajectories in the phase space and their topology. The time-domain information about the motion usually discards transients and covers the steady state, since the Fourier transform of a nonstationary oscillation has little physical sense.

Nevertheless, nonlinear transients are present on a variety of real-world problems. In the field of mechanic engineering, they can arise in the interaction of motors and structures. Acoustics has multiple problems that present nonlinear transients, such as transducers' dynamics, musical instruments' analysis, as well as sound perception and bioacoustics. Transients of electronic circuits, which can be nonlinear, have a potential interest in areas such as signal transmission. It should also be remembered that, in a semiclassical approach, most problems of interaction of radiation and matter deal with nonlinear oscillators' transients. These and many other phenomena cannot be fully understood by a steady state approach.

In the present work, however, we focus the attention precisely on the transient region. The Gabor transform technique was chosen to examine the spectrum evolution of a nonlinear system through the transient. This technique has been previously used by the authors to analyze transients of a double potential well free oscillator [6]. In this article, we describe the "transient spectrum" or spectrogram of a forced oscillator.

The spectrograms of the driven oscillator shows temporary quasi-periodic oscillations. Although they are transient phenomena, they bear a close resemblance with steady-state quasiperiodic oscillations. There are also important differences that we shall point out later. Before we proceed, let us briefly recall the basic definitions of steady state oscillations

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Sinusoidally driven nonlinear oscillators can, in certain cases, exhibit quasi-periodic behavior [5]. In this regime, the spectrum has a discrete set of peaks, and all present frequencies can be expressed as sums and differences of a countable set of basis frequencies f_j and their harmonics. For a p-periodic oscillation, the allowed frequencies are

$$f_i = \sum_{j=1}^{p} k_j f_j, \qquad (1)$$

where the k_j are integers. In other words, the frequencies obeys a sort of Ridberg–Ritz "combination principle".

In the case of two oscillating external forces, the basis frequencies are the same in the driving force and their subharmonics. The power spectrum contains then the so-called combination tones. In a system with an autonomous limit cycle, such as the van der Pol oscillator, the external frequency and the limit cycle frequency constitute the basis of the quasiperiodic oscillations. These are examples of two-periodic motions, and the resulting frequencies are the harmonics of the basis frequencies, as well as their sums and differences.

In this article, we show that the spectrogram of the transient region of a forced nonlinear oscillator shows series of evenly spaced peaks as a quasiperiodic oscillator. Each "time slice" of the spectrogram is equivalent to a three-periodic motion. As expected, one of the basis frequencies is the external force frequency. The 1/3 subharmonic of the driving force is also present in the basis. However, the third basis frequency is not equal nor commensurable to any characteristic system frequency. It is a function of the external force frequency and magnitude, and also the form of the system potential well. As the system approaches the steady state, these lines decrease in magnitude at different rates, so that each line has its own "*lifetime*".

2. Transient analysis

The system chosen to the transient analysis is the Duffing oscillator, defined by the following DE,

$$m\ddot{x} + c\dot{x} - k_1 x - k_2 x^2 - k_3 x^3 = F_0 \cos(\omega t). \qquad (2)$$

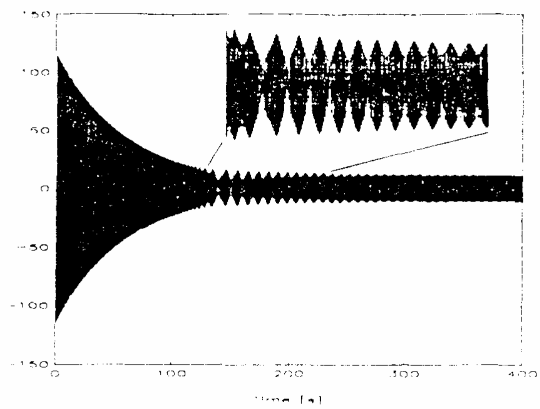

Fig. 1. Transient of the forced Duffing oscillator.

The oscillator's parameters are $m = 1$, $k_1 = 4\pi^2$, $k_2 = 0$, $k_3 = 4\pi^2$ and $c = 0.025$. The potential curve has the form of a single symmetric potential well, with a small oscillation frequency 1 Hz. The external force has a strength $F_0 = 4$ and a frequency 1.5 Hz. The damping constant c is chosen so that the relative energy dissipation per cycle is small. The initial position is $x_0 = 0$ and the velocity $v_0 = 120$ m/s. We have employed in all numerical simulations the DGEAR routine, which implements the Adams–Moulton algorithm with polynomials up to 12th order. The transient obtained using the above parameters is shown in Fig. 1.

The picture actually shows the envelope, since individual cycles are too small in the graphic scale. After a steady energy decay, the system undergoes transient amplitude modulations (AM) which eventually vanish. We analyze here this second phase of the transient.

The Gabor transform [7,8] is implemented through a dedicated code written in the MATLAB® language and using direct calculation from definition and Morlet wavelets [9], which have the form of Gaussian wave packets, as basis elements. The Gabor transform $G_s(b, \omega)$ of a function $s(T_j)$, a sampled version of a continuous time function $s(t)$, reads then

$$G_s(b, \omega) = \langle s(t), g_{(b,\omega)}(t) \rangle = \sum_{k=1}^{N} s(T_k)\, g^*_{(b,\omega)}(T_k). \qquad (3)$$

where

$$g_{(b,\omega)}(t) = \exp[\mathrm{i}\,\omega(t-b)]\,\exp\left(\frac{t-b}{2\sigma_t}\right)^2. \qquad (4)$$

In words, the Gabor transform coefficients are obtained through the Hilbertian scalar product between the basis function and the analyzed function. The sampling rate should, of course, respect the limits imposed by the Nyquist theorem [10].

Both numerical simulation and Gabor analysis are performed using floating point double precision numbers with 16 digit mantissa and 3 digit exponent. The Gabor transform maps the original function $s(t)$ onto the time–frequency phase space. The time mean square deviation of the Morlet wavelets of Eq. (4) is $\sigma_t = 100/\sqrt{2\pi} = 39.86$ s. This peculiar value comes from the choice of the frequency resolution, linked with σ_t through the uncertainty relations for Gaussian wave packets. We use only the square modulus of the transform, $|G(b, \omega)|^2$, which can be interpreted as an energy density [11] and is also called Gabor spectrogram.

We show in Fig. 2 a typical spectrogram of the nonlinear transient. It can be read as a chart of the power spectral density in the time–frequency domain. We use a gray scale with a density proportional to the logarithm of the spectral power. The main line at 1.5 Hz corresponds, of course, to the external frequency. Surrounding this line, there are sidebands, constituted by evenly spaced lines. If the system has simple AM oscillations, there would be just a single pair of extra lines. The oscillations can, however, be identified as simultaneous frequency and amplitude modulation (AM–FM). The modulation frequency is identical to the third basis element of the quasi-periodic oscillations mentioned above.

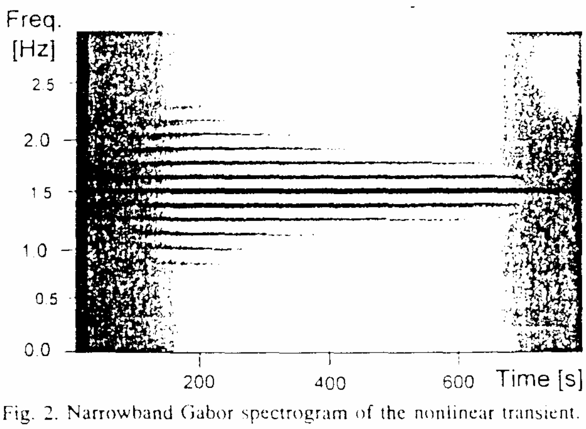

Fig. 2. Narrowband Gabor spectrogram of the nonlinear transient.

3. A model for the sidebands structure

The modulated oscillations can be interpreted as follows: The nonlinear oscillator and the external force exchange energy back and forth, so that the system energy oscillates during the transient. The damping term ensures the approach to an equilibrium state, while energy oscillations decrease. The frequency of these oscillations can be estimated through a simple model involving energy balance considerations. Only the main hypotheses of the derivation are sketched here.

We approximate the system motion to a sinusoidal function of time, so that only the coupling between the first harmonic of the motion and the external force is taken on account. Moreover, we assume slowly varying energy and frequency. The energy transfer rate between the oscillator and the external force is then estimated using a "power factor" similar to the one used in electric circuits. We assume also a small phase departure from steady state condition, and obtain the following estimation of the modulation frequency,

$$f_m = \omega_m/2\pi,$$

$$\omega_m \simeq \left(\frac{1}{2}\frac{\mathrm{d}\omega(E)}{\mathrm{d}E}\bigg|_{\omega_0} F_0\, v_{max}(\omega_0)\right)^{1/2} \qquad (5)$$

The function $\omega(E)$ is the angular frequency of the free oscillator, as a function of the total energy E. The derivative of this frequency is evaluated for the free oscillator at the frequency ω_0 of the external forc[illegible] F_0 is the magnitude of the external sinusoidal force, as in Eq. (2), while $v_{max}(f_0)$ is the maximum velocity of the free oscillator when it oscillates at the same fr[illegible]uency f_0 as the external force. These functions de[illegible]d on the detailed form of the potential well, an[illegible]ire a separate numerical evaluation.

Due [illegible] oscillating energy exchange, the system m[illegible] both amplitude and frequency modulated. T[illegible]tem oscillations can be expanded over a set of sinusoidal oscillations, yielding the spectrum of an AM–FM wideband modulated oscillation [12],

$$x(t) = A_c \sum_{n=-\infty}^{\infty} \{J_n(\beta) + \tfrac{1}{2}m_a[J_{n-1}(\beta) + J_{n+1}(\beta)]\} \times \exp[i(\omega_c + n\omega_m t)], \quad (6)$$

where J_n are Bessel functions of first kind and ω_c, the "carrier frequency", is identified with ω_0, the external force frequency. This expression shows that the AM–FM oscillations appear in the frequency domain as a cluster of evenly spaced sharp peaks, with magnitudes decreasing as n increases. The parameter β indicates the modulation depth. In our problem, this parameter decreases exponentially in time, while the system approaches the steady state. The energy oscillation model above fits well the line spacing observed on the spectrograms with the external force F_0 ranging from 0.5 to 10.

The spectrogram shown in Fig. 2 expands the first harmonic region only, covering the range from 0 to 3 Hz. There are, however, similar "multiplet-like" structures at higher frequencies. We show in Fig. 3 a "time slice" of the Gabor transform of the oscillations, which is roughly a "local spectrum". There are clusters of evenly spaced lines surrounding the frequencies of the odd harmonics of the sinusoidal external force. The potential symmetry "forbids" the even harmonics that would be present if the quadratic term of Eq. (2) were not zero.

Clusters of evenly spaced peaks are also visible around the frequency $f_0/3$, a subharmonic of the driving force (visible at the bottom of Fig. 2). Moreover, they are present around the frequencies $(n - 1/3)F_0$, so that this subharmonic is also a basis of the quasiperiodic movement. Some of the lines are labeled with the integer numbers k_j of Eq. (1). The first integer is the coefficient of the frequency f_0, the second, of the subharmonic $f_0/3$ and the third multiplies the modulating frequency f_m.

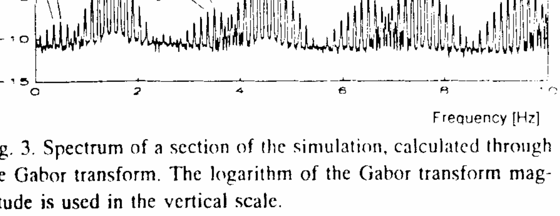

Fig. 3. Spectrum of a section of the simulation, calculated through the Gabor transform. The logarithm of the Gabor transform magnitude is used in the vertical scale.

4. Conclusions

The Gabor transform can map the spectrum of transient nonlinear vibrations. We examine the synchronization process of a forced oscillator with a cubic nonlinearity. The spectrograms generated by the Gabor transform shows clusters of evenly spaced lines, a signature of quasi-periodic oscillations. The structure of the line clusters corresponds to a three-periodic movement. One of the basis frequencies present is incommensurable with the external force frequency. These spectra are identified with simultaneous amplitude and frequency modulation of the system. A simple model, based on energy transfer considerations, can explain the AM–FM behavior and correctly calculate the modulating frequency.

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